NORMAL APPLICATIONS

Unit 3B: Random Variables

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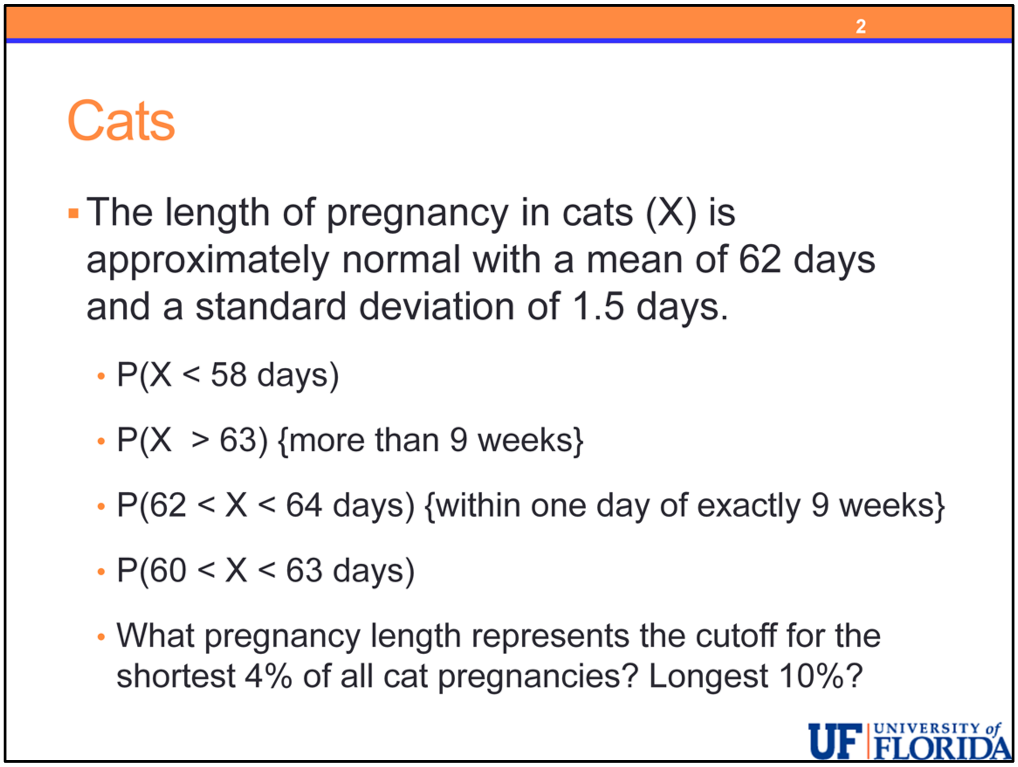
Let's go through some examples of applying the normal distribution in practice.



We will work with gestation period of domestic cats.

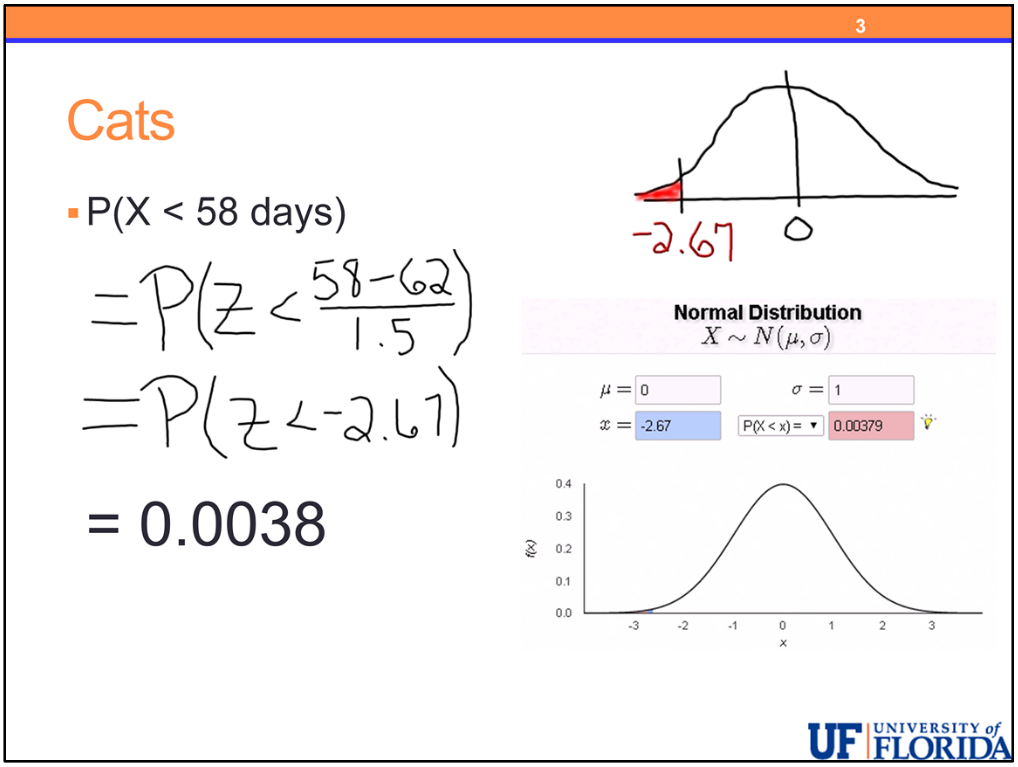
Suppose that the length of pregnancy in cats (which we will denote X) is approximately normal with a mean of 62 days and a standard deviation of 1.5 days.

We want to find the following probabilities:

> P(X < 58 days)
> P(X > 63) {this is more than 9 weeks}
> P(62 < X < 64 days) {this is within one day of exactly 9 weeks}
> P(60 < X < 63 days)

And, in reverse:

> What pregnancy length represents the cutoff for the shortest 4% of all cat pregnancies?
> What pregnancy length represents the cutoff for the longest 10% of all cat pregnancies?



To find the probability that X is less than 58 days, first we convert the X to its corresponding Z-score by subtracting the mean, 62 and dividing by the standard deviation of 1.5.

We get P(Z < (58-62)/1.5) = P(Z < -2.67)

At this point I suggest sketching the situation as we have in the upper right corner. It isn't exactly accurate as we can see by comparison to the actual solution given by the calculator but it helps make sure you end up calculating what you hope if you have the sketch to review.

Here we illustrate the solution with the Non-JAVA online calculator.

We type the z-score of -2.67 into the box for X, making sure that the mean and standard deviation are still set to zero and one.

For this calculator we also need to decide whether to calculate the probability below or above. I will choose below always to illustrate the solution in the same way as using the table.

This gives an area to the left of -2.67 of 0.00379 or, rounding to 4 decimal places, 0.0038.

Table

Table entry for z is the area under the standard normal curve to the left of z.

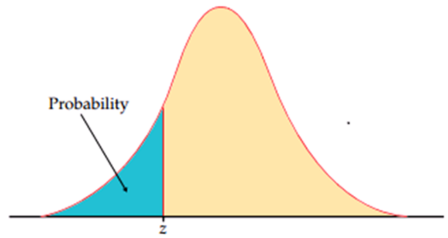


TABLE A

Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064

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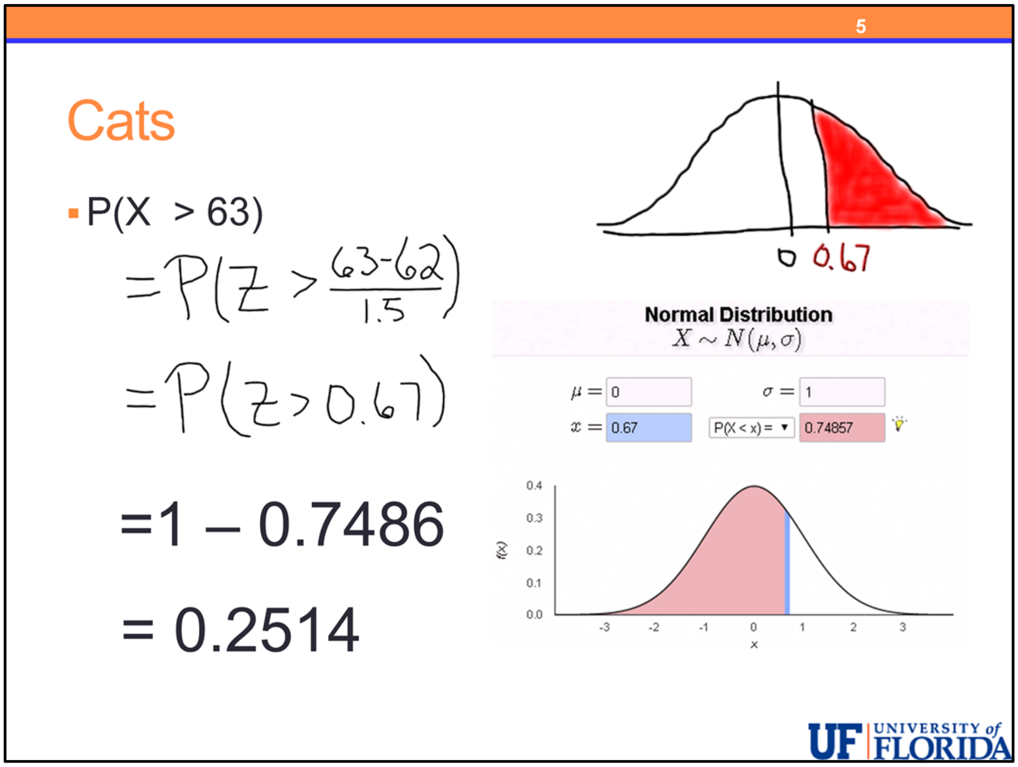
We can also use the normal table to find the area. This table always gives the area to the left and there is one page for negative z-scores – shown here and another for positive z-scores.

We find the first two digits of the z-score on the left in the column labeled z. Here we find negative 2.6 and then we find the last digit on the top, here 0.07 since the z-score is negative 2.6**7**.

Finding the intersection of the -2.6 row with 0.07 column we find an area of 0.0038 again.



To find the probability that X is greater than 63 days, first we convert the X to its corresponding Z-score by subtracting the mean, 62 and dividing by the standard deviation of 1.5.

- We get P(Z < (63-62)/1.5) = P(Z < 0.67)
- We have our sketch in the upper right.
- Here we illustrate the solution with the Non-JAVA online calculator.
- We type the z-score of 0.67 into the box for X, making sure that the mean and standard deviation are still set to zero and one.
- Again I will choose below always to illustrate the solution in the same way as using the table.

This gives an area to the left of 0.67 of 0.7486 rounded to 4 decimal places.

However we need the area to the right as we want the probability Z is greater than 0.67, so we need to subtract the area 0.7486 from 1 to get a final area of 0.2514.

Notice that if we had forgotten to subtract from 1, we would have 0.7486 which does not make sense from our original sketch. The picture in the calculator does not match our sketch since it found the area to the left and we want the area to the right.

You are welcome to set the tool to P(X > x) and skip the subtraction if you wish.

Table

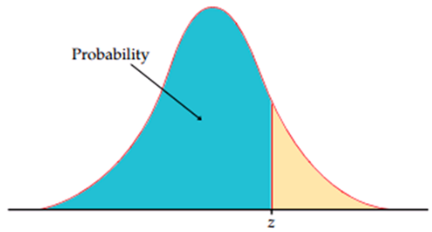


Table entry for *z* is the area under the standard normal curve to the left of *z*.

TABLE A

Standard normal probabilities (continued)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015

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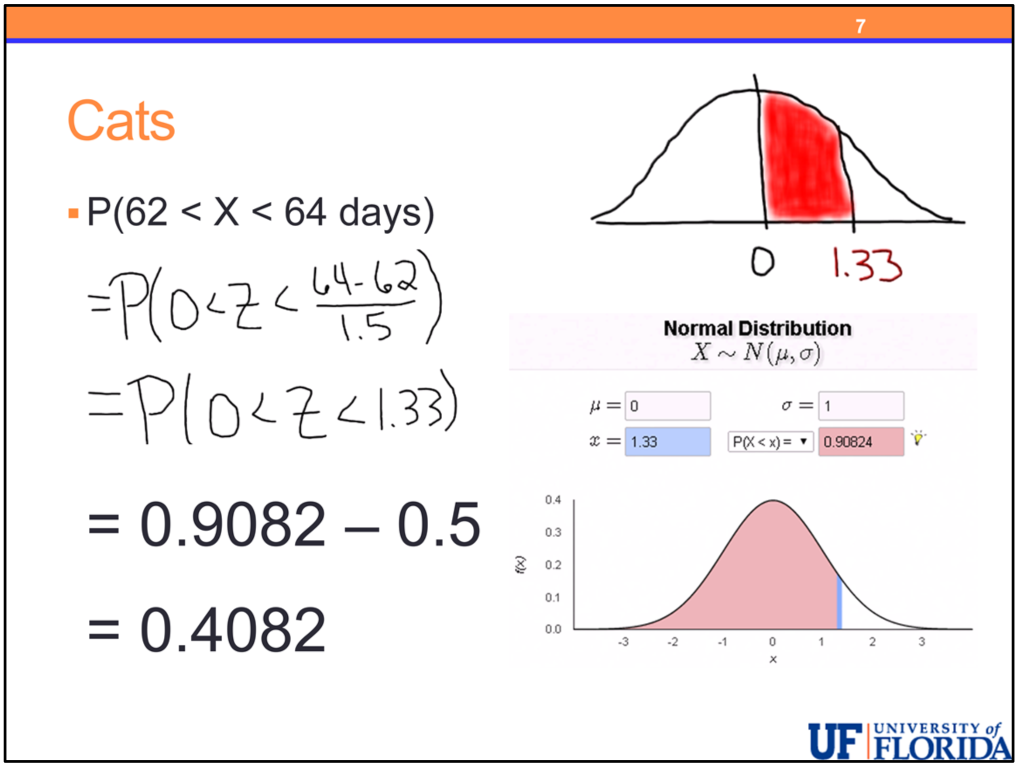
We can also use the normal table to find the area. This table always gives the area to the left and there is one page for negative z-scores and another for positive z-scores – shown here.

We find the first two digits of the z-score on the left in the column labeled z. Here we find 0.6 and then we find the last digit on the top, here 0.07 since the z-score is 0.6**7**.

Finding the intersection of the 0.6 row with 0.07 column we find an area of 0.7486 again as the area to the left, then we subtract from 1 to get our previous answer of 0.2514.



Now we have a between problem – the probability that X is between 62 and 64.

When we convert the lower value of 62 to a z-score, we get zero, since 62 is the mean. Not all problems will be that way as we will see shortly.

For the upper value of 64, we get Z = (64 – 62)/1.5 = 1.33.

We have again sketched the picture in the upper right. Notice that the area below 0 is automatically known to be 0.5.

Using the online calculator to find the area to the left for Z = 1.33 we get 0.9082. This is everything from 1.33 and below. We need to subtract the area to the left of zero which is 0.5 to get 0.9082 – 0.5 = 0.4082.

Checking this against our original sketch this seems reasonable.

Table

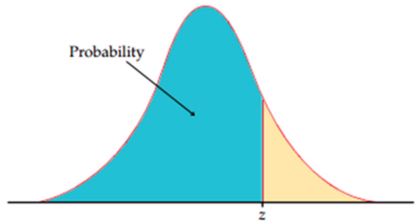


Table entry for *z* is the area under the standard normal curve to the left of *z*.

TABLE A

Standard normal probabilities (continued)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

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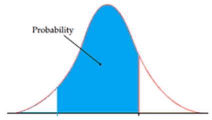
Using the table, we find the first two digits of the z-score on the left 1.3 and then we find the last digit on the top, here 0.03 since the z-score is 1.3**3**.

Finding the intersection of the 1.3 row with 0.03 column we find an area of 0.9082 again.

Cats



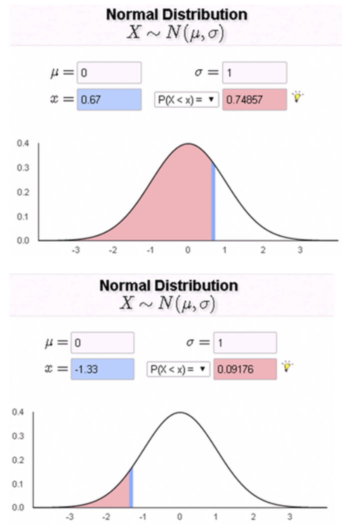
- P(60 < X < 63 days)

$$= P\left(\frac{60-62}{1.5} < X < \frac{63-62}{1.5}\right)$$

$$= P(-1.33 < X < 0.67)$$

= 0.7486 – 0.0918

= 0.6568



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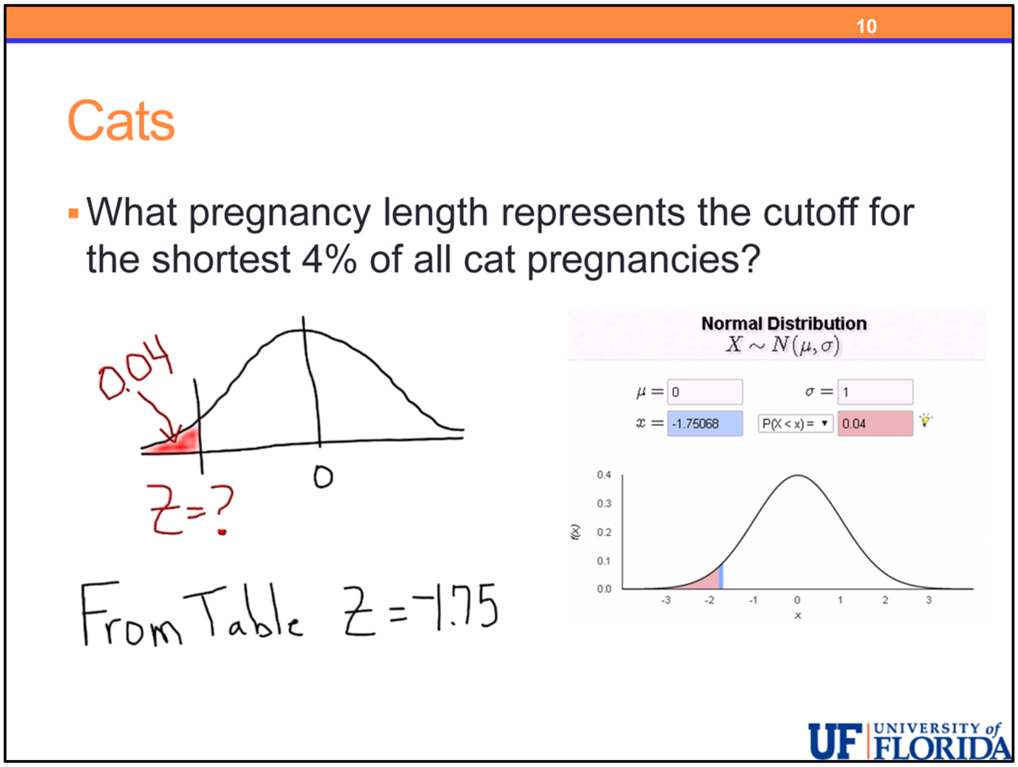
Now the probability that X is between 60 and 63. We found the z-score for 63 earlier, it was 0.67.

For X = 60 we have Z = -1.33.

Sketching this we will need to find the area to the left of Z = 0.67 and then subtract from it the area to the left of Z = -1.33.

For 0.67 we get 0.7486 and for -1.33 we get 0.0918 and subtracting gives the answer of 0.6568.



Now we turn the process around. We ask, what pregnancy length corresponds to the shortest 4% of cat pregnancies.

We can sketch this picture by shading a small area on the lower left. We are looking for the z-score instead of the probability.

This a two step process. First we find the Z-score associated with an area to the left of 4% and then we will convert that to an X-value representing the length of cat pregnancy.

To use the online calculator to find the z-score, we simply type 0.04 in the probability box and hit enter.

Rounding the z-score to two decimal places we get -1.75.

Table

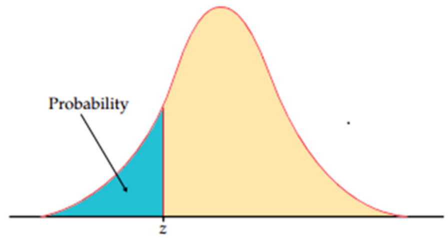


Table entry for *z* is the area under the standard normal curve to the left of *z*.

TABLE A

Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611

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To find the Z-score using the normal table we have to start by scanning the interior of the table looking for the **closest** value to 0.04. Be sure to find the closest value.

Here we find 0.0401 is the closest to 0.04. Following this back to the left side we find Z = negative 1.7 for the first two digits and following up we find 0.05 as the last digit giving a Z-score of negative 1.75.

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Cats

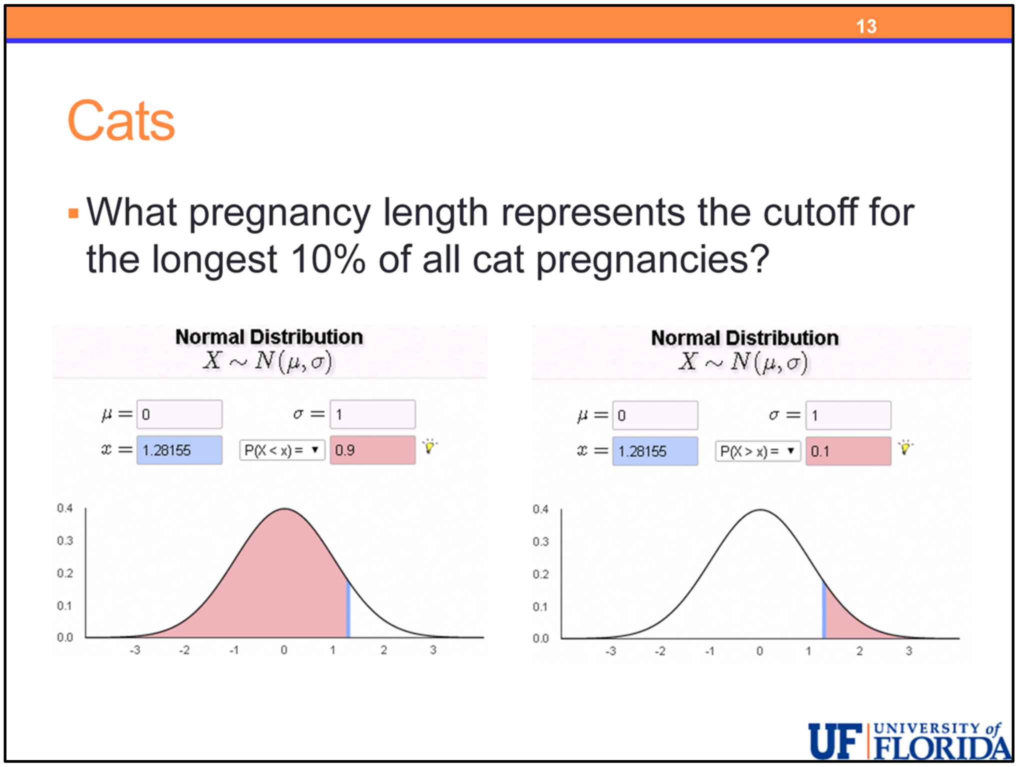
- What pregnancy length represents the cutoff for the lowest 4% of all cat pregnancies?

$$X = \mu + Z\sigma$$
$$= 62 + (-1.75)(1.5)$$
$$= 59.4 \text{ days}$$

To convert this back to a cat pregnancy length, we use the equation X = mu + Z times sigma.

Substituting 62 for mu, -1.75 for Z, and 1.5 for sigma we get about 59.4 days.

So only 4% of cat pregnancies last 59.4 days or less.



Here if we are using the table or the JAVA version of the calculator, then we need to realize that the cutoff for the top 10% is equivalent to the bottom 90% or the 90th percentile.

We can enter 0.9 in the online calculator to find a z-score of 1.28.

Alternatively, on the right we show using the non-JAVA version, choosing P(X > x) and entering 0.1 directly. We can see the online calculator shading the region used which can help to make sure you are making the correct choice.

Table

Table entry for *z* is the area under the standard normal curve to the left of *z*.

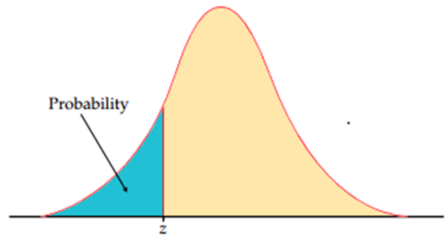


TABLE A

Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441

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For the table, we need to find the area to the left of 0.9.

To find the Z-score using the normal table we have to start by scanning the interior of the table looking for the **closest** value to 0.90. Be sure to find the closest value.

Here we find 0.8997is the closest to 0.90. Following this back to the left side we find Z = 1.2 for the first two digits and following up we find 0.08 as the last digit giving a Z-score of 1.28.

15

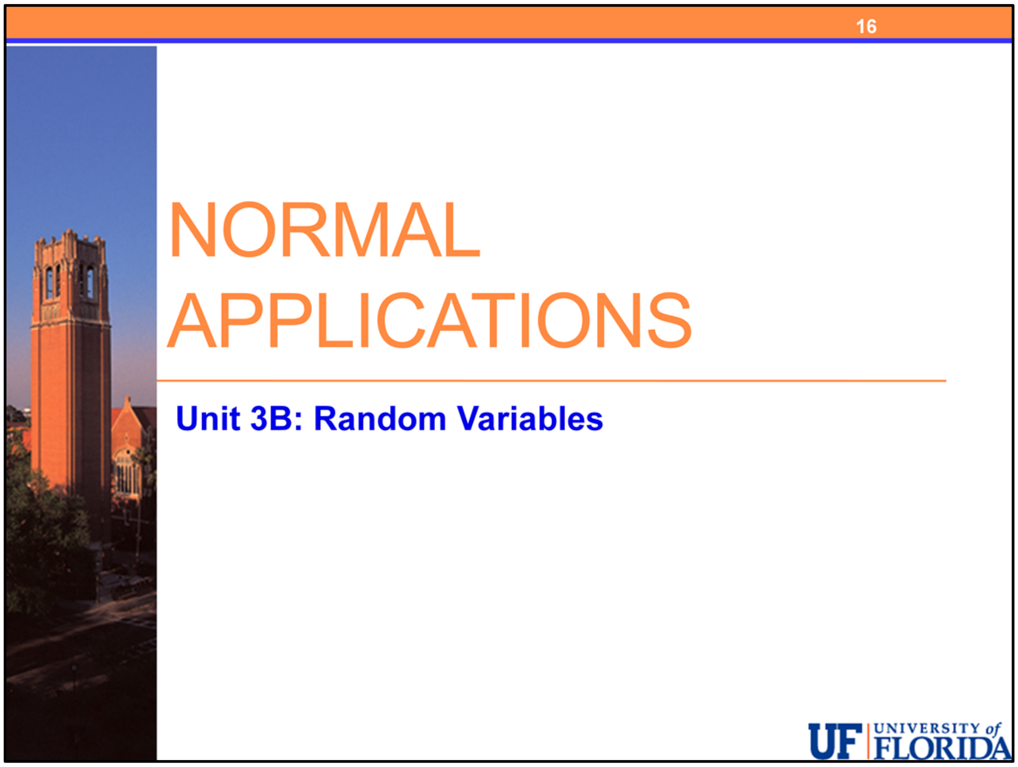
Cats

- What pregnancy length represents the cutoff for the longest 10% of all cat pregnancies?
- $X = \mu + z\sigma$
- $= 62 + (1.28)(1.5)$
- $= 63.9$

To convert this back to a cat pregnancy length, we use the equation X = mu + Z times sigma.

Substituting 62 for mu, 1.28 for Z, and 1.5 for sigma we get about 63.9 days.

So only 10% of cat pregnancies last 63.9 days or more.



Finding probabilities such as these is useful in it's own right for many situations but we will soon see that this skill plays a part in sampling distributions, confidence intervals, and hypothesis tests.