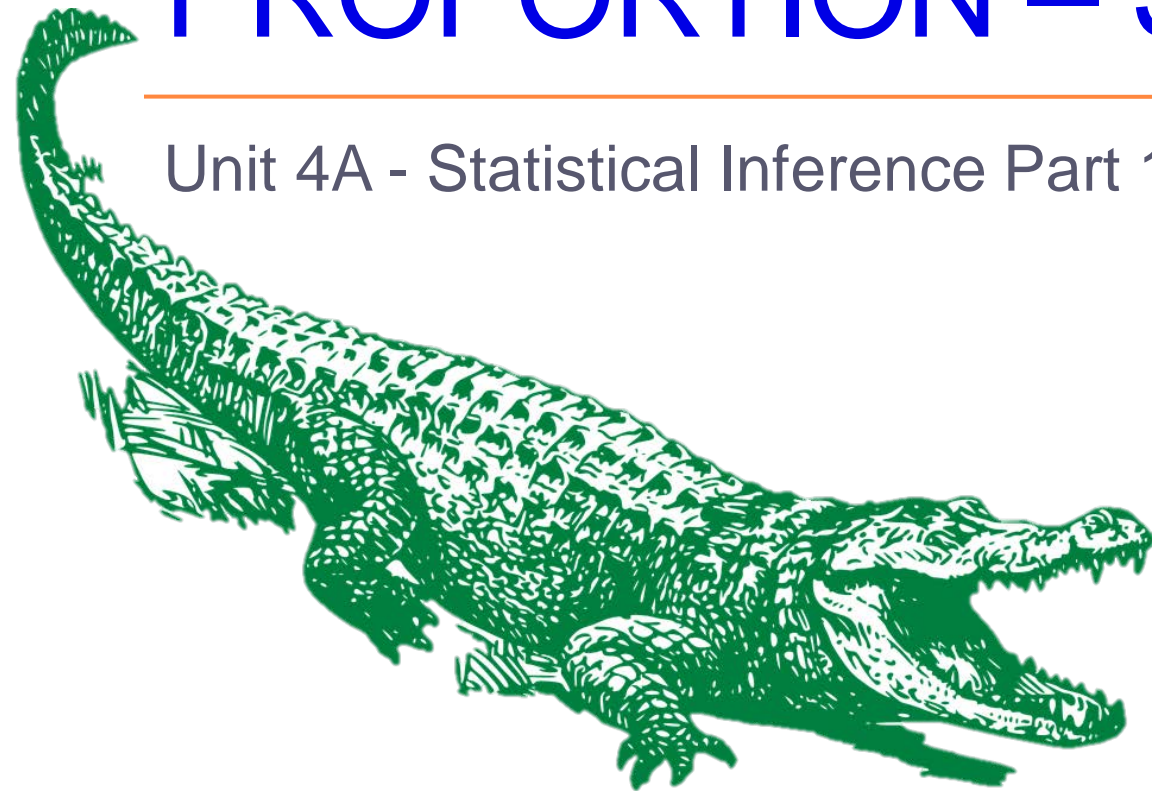
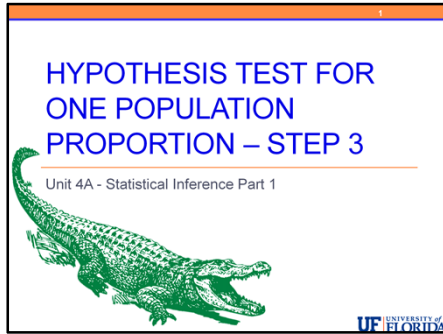




HYPOTHESIS TEST FOR ONE POPULATION PROPORTION – STEP 3

Unit 4A - Statistical Inference Part 1





Now let's look at STEP 3 for the **z-test for the population proportion (p)**.

Here we

- Use our test statistic to
- Determine the p-value of our test

The p-value is the probability of observing data as or more extreme than our data assuming the null hypothesis is true.

This test for one proportion will be the only situation where we will find the p-value ourselves using normal probabilities. For all remaining tests in this course, we will rely on software to obtain the p-values for us.

We are using this situation to help us understand p-values so that we will be more able to correctly use and understand the results we obtain from software for future tests.



TEST STATISTIC vs. P-VALUES

■ TEST STATISTIC

- The **LARGER** the test statistic,
- The **further p-hat is from p_0**
- And the **MORE EVIDENCE AGAINST H_0**


■ P-VALUE

- The **SMALLER** the p-value,
- The **further p-hat is from p_0**
- And the **MORE EVIDENCE AGAINST H_0**

2

TEST STATISTIC vs. P-VALUES

- **TEST STATISTIC**
 - The **LARGER** the test statistic,
 - The further $p\text{-hat}$ is from p_0
 - And the **MORE EVIDENCE AGAINST H_0**
- **P-VALUE**
 - The **SMALLER** the p-value,
 - The further $p\text{-hat}$ is from p_0
 - And the **MORE EVIDENCE AGAINST H_0**



Both the test statistic and the p-value are measures of the evidence against H_0 .

Although conclusions can be drawn directly from test statistic values, in practice most researchers use the p-value along with the 5% significance level to determine whether or not to reject the null hypothesis.

For **TEST STATISTICS**

The **LARGER** the test statistic,
 The **further $p\text{-hat}$ is from p_0**
 And the **MORE EVIDENCE AGAINST H_0**

But for **P-VALUES** it is the opposite.

The **SMALLER** the p-value,
 The **further $p\text{-hat}$ is from p_0**
 And the **MORE EVIDENCE AGAINST H_0**

One benefit of using p-values is that the values of test statistics may not always be as interpretable and easy to work with as z-scores which follow a normal distribution. P-values have the same interpretation for every test and therefore we can develop a unified approach to STEPS 3 and 4 for all tests regardless of the results from STEP 2.



P-VALUES

- The **P-VALUE** is the
 - **probability** of
 - observing a **test statistic** as extreme as that observed (or even more extreme)
 - **assuming** that the **null hypothesis is true**
- By “**extreme**” we mean extreme **in the direction(s) of the alternative hypothesis**.

3

P-VALUES

- The **P-VALUE** is the
 - probability of
 - observing a **test statistic** as extreme as that observed (or even more extreme)
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Intuitively, the p-value is the **probability** of observing **data like those observed** assuming that H_0 is true.

Let's be a bit more formal:

- Since this is a probability question about the **data**, it makes sense that the calculation will involve the data summary, the **test statistic**.
- What do we mean by “**like**” those observed? By “like” we mean “**as extreme or even more extreme.**”
- Putting it all together, we get that in **general**:

The **p-value** is the **probability of observing a test statistic as extreme as that observed (or even more extreme) assuming that the null hypothesis is true.**

By “**extreme**” we mean extreme **in the direction(s) of the alternative hypothesis**.

If our alternative hypothesis is that a particular treatment will result in weight loss but instead it results in weight gain, this would NOT count as evidence in support of our alternative – this may be extreme but if so, it is NOT in the direction of the alternative hypothesis, it is in the opposite direction.

Now, we will formalize this for each of our possible alternative hypotheses.


Null Distribution

- Test Statistic:
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

- If our conditions are satisfied:

$$np_0 \geq 10$$

- Random Sample
- Sample Size Large Enough

$$n(1 - p_0) \geq 10$$


- Then, assuming H_0 is true, this test statistic is distributed as a **STANDARD NORMAL DISTRIBUTION**
- We use this distribution to find our p-values!**

Null Distribution

- Test Statistic: $z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
- If our conditions are satisfied:
 - Random Sample
 - Sample Size Large Enough $\rightarrow np_0 \geq 10$ and $n(1-p_0) \geq 10$
- Then, assuming H_0 is true, this test statistic is distributed as a **STANDARD NORMAL DISTRIBUTION**
- We use this distribution to find our p-values!

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Recall our test statistic equation which takes the form of a z-score for p-hat.

From our work on sampling distributions, if our two conditions are satisfied – having a random sample and a large enough sample relative to the hypothesized proportion,

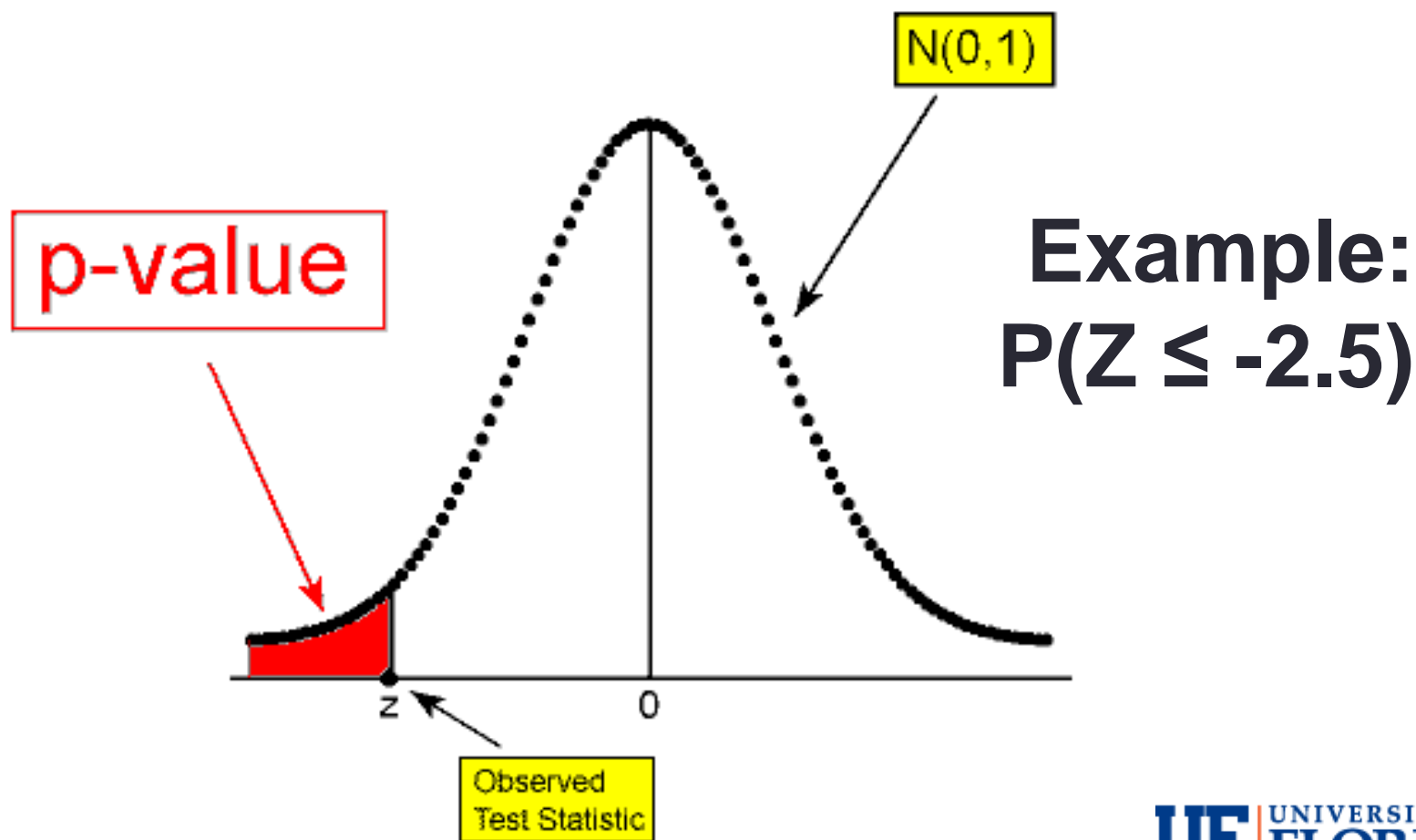
then under the assumption that the null hypothesis is true our test statistic is distributed as a standard normal distribution and we can use this distribution, which we have worked with already using z-scores, to calculate p-values.

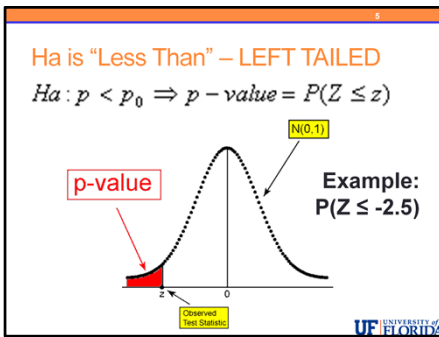
Notice that we went through similar arguments in our discussion about confidence intervals but there we used the normal distribution to find cutoffs for the appropriate confidence level. Here we will use the normal distribution to find probabilities.

Maybe you can see now why we needed to learn about normal distributions and be able to work with them, we needed it here to be able to discuss p-values and we needed it for confidence intervals to find the cutoffs required.

Ha is “Less Than” – LEFT TAILED

$$H_a : p < p_0 \Rightarrow p\text{-value} = P(Z \leq z)$$





If the alternative hypothesis is $H_a: p < p_0$ (**less than**), then "extreme" means **small or less than**, and the p-value is:

- The probability of observing a test statistic **as small as that observed or smaller** if the null hypothesis is true.

We can write this in notation as $P(Z \leq \text{our test statistic})$ which here we denote by the lower case z . This notation is sometimes confusing for students. Here the capital Z represents the DISTRIBUTION theoretically – the random variable. And the lower case z represents the value of our test statistic.

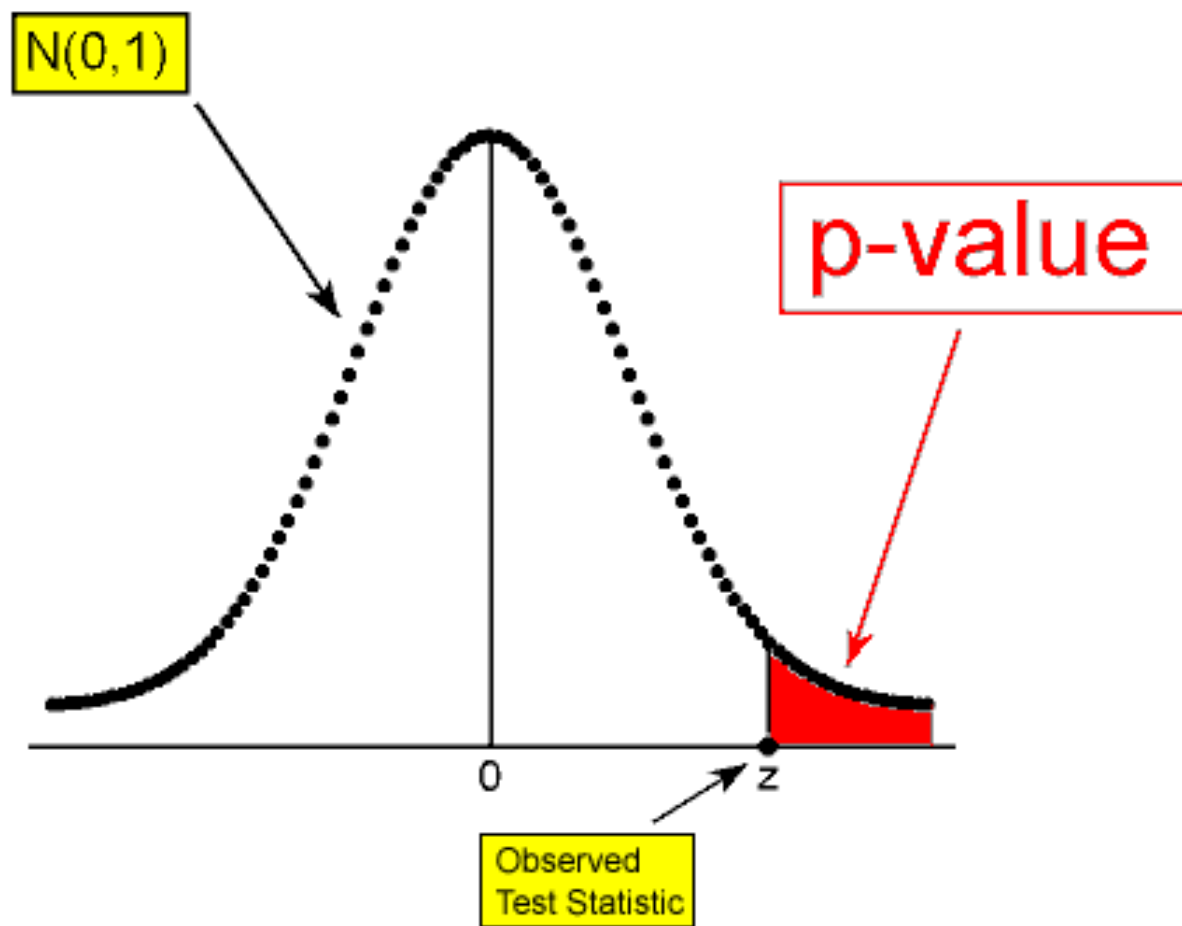
Hopefully when we look at our examples this will be clear.

You do not need to use this notation but you should start with writing the probability you are calculating with your test statistic instead of lower case z .

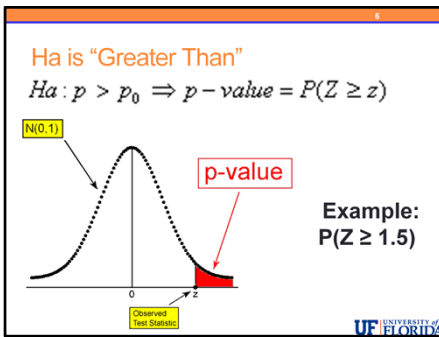
So, if you have a test statistic of $z = -2.5$ then we would write $P(Z \leq -2.5)$

H_a is “Greater Than”

$$H_a : p > p_0 \Rightarrow p\text{-value} = P(Z \geq z)$$



Example:
 $P(Z \geq 1.5)$



If the alternative hypothesis is $H_a: p > p_0$ (**greater than**), then "extreme" means **large or greater than**, and the p-value is:

- The probability of observing a test statistic **as large as that observed or larger** if the null hypothesis is true.

We can write this in notation as $P(Z \geq \text{our test statistic})$ which again we denote by the lower case z.

For example, if our test statistic is 1.5, we would write $P(Z \geq 1.5)$

Ha is “Not Equal To”

p-value

$$H_a : p \neq p_0 \Rightarrow$$

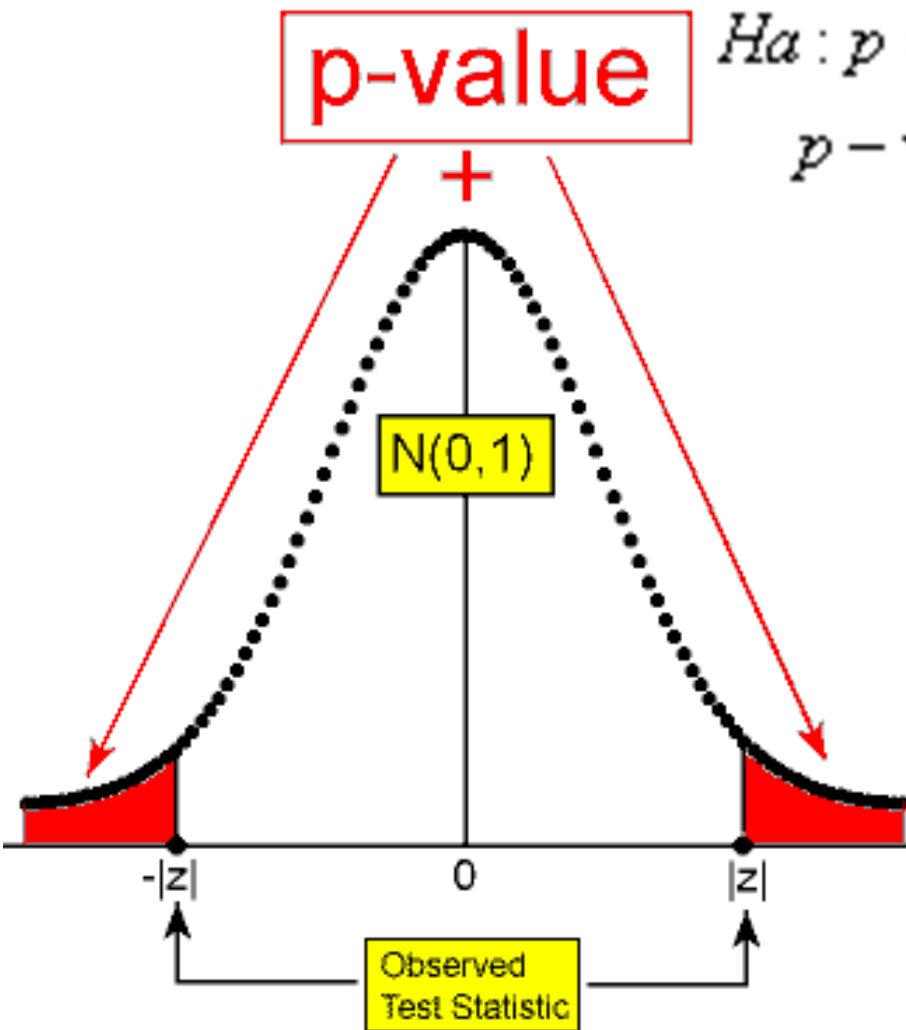
$$p\text{-value} = P(Z \leq -|z|) + P(Z \geq |z|) \\ = 2P(Z \geq |z|)$$

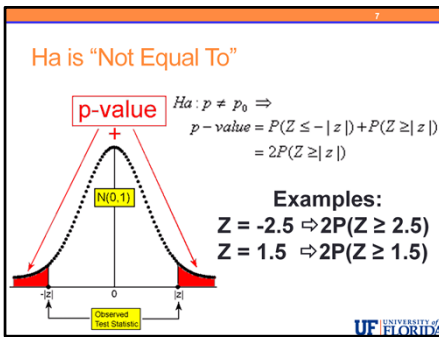
N(0,1)

Examples:

$$Z = -2.5 \Rightarrow 2P(Z \geq 2.5)$$

$$Z = 1.5 \Rightarrow 2P(Z \geq 1.5)$$





If the alternative is $H_a: p \neq p_0$ (**different from**), then "extreme" means extreme in either direction **either small or large (i.e., large in magnitude) or just different from**, and the p-value therefore is:

- The probability of observing a test statistic **as large in magnitude as that observed or larger** if the null hypothesis is true.

Here are two examples:

If $z = -2.5$:

- The p-value = probability of observing a test statistic as small as -2.5 or smaller or as large as 2.5 or larger.

If $z = 1.5$:

- The p-value = probability of observing a test statistic as large as 1.5 or larger, or as small as -1.5 or smaller.

The easiest way to find the p-value in this case is to take the absolute value of our test statistic – in other words – remove any negative signs and then find the right-tailed probability: $P(Z \geq \text{the absolute value of our test statistic})$ and then double this value.

Our general formula for the p-value in this case is 2 times $P(Z \geq \text{the absolute value of our test statistic})$.

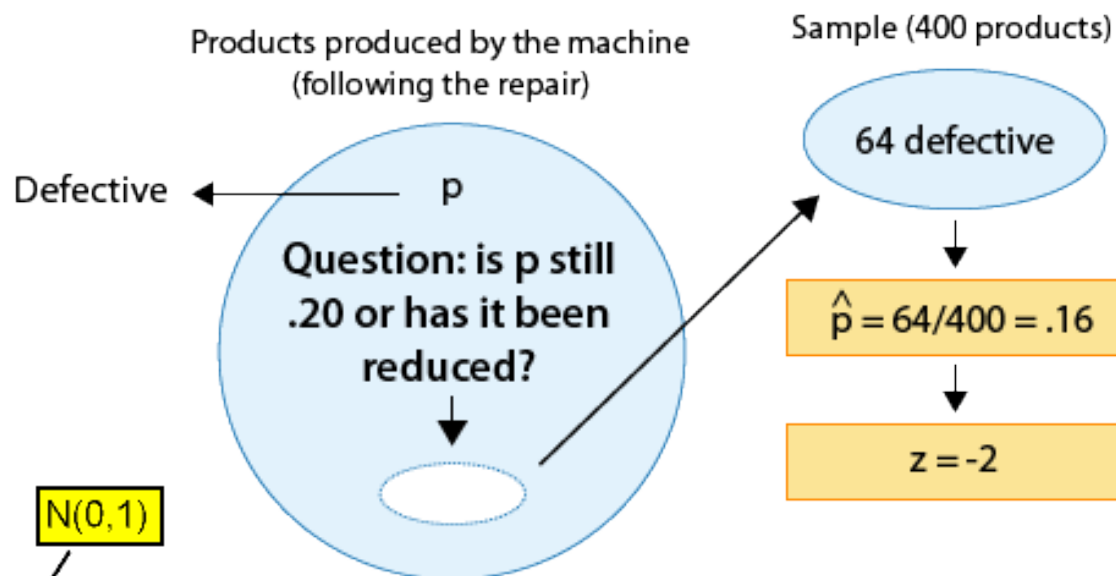
No matter how you find this p-value, you need to make sure that you obtain both the upper and lower tails. You can do this by

- Finding both probabilities and adding them together – you will likely tire of this method as they are the same value so it is easier to
- Pick either tail – but only one of them and then find the probability, and finally double the result.

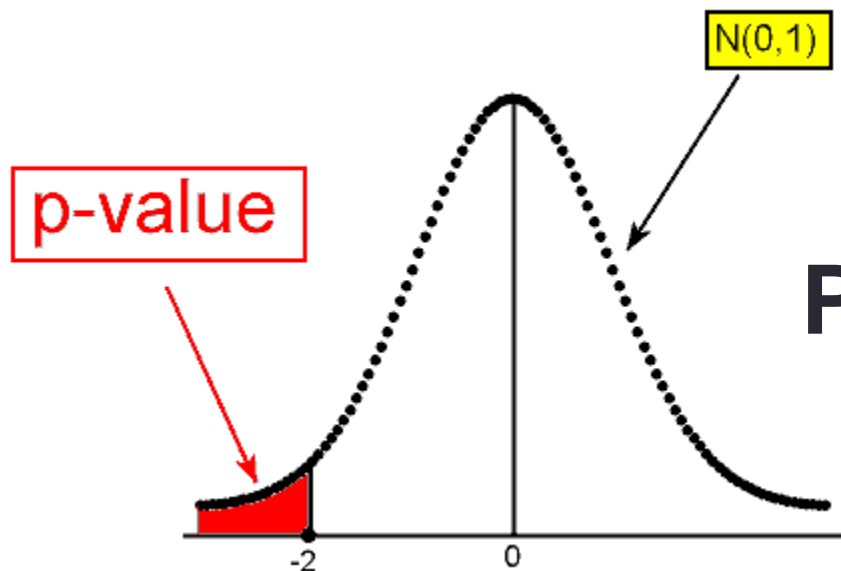
Now let's return to our examples.

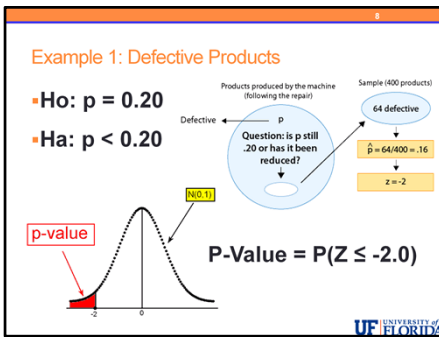
Example 1: Defective Products

- $H_0: p = 0.20$
- $H_a: p < 0.20$



$$P\text{-Value} = P(Z \leq -2.0)$$





Has the proportion of defective products been reduced as a result of the repair?

The p-value in this case is:

- The probability of observing a test statistic as small as -2 or smaller, assuming that H_0 is true.

OR (recalling what the test statistic actually means in this case),

- The probability of observing a sample proportion that is 2 standard errors or more below the null value ($p_0 = 0.20$), assuming that p_0 is the true population proportion.

OR, more specifically,

- The probability of observing a sample proportion of 0.16 or lower in a random sample of size 400, when the true population proportion is $p_0 = 0.20$

The p-value is found as shown in the normal curve here.



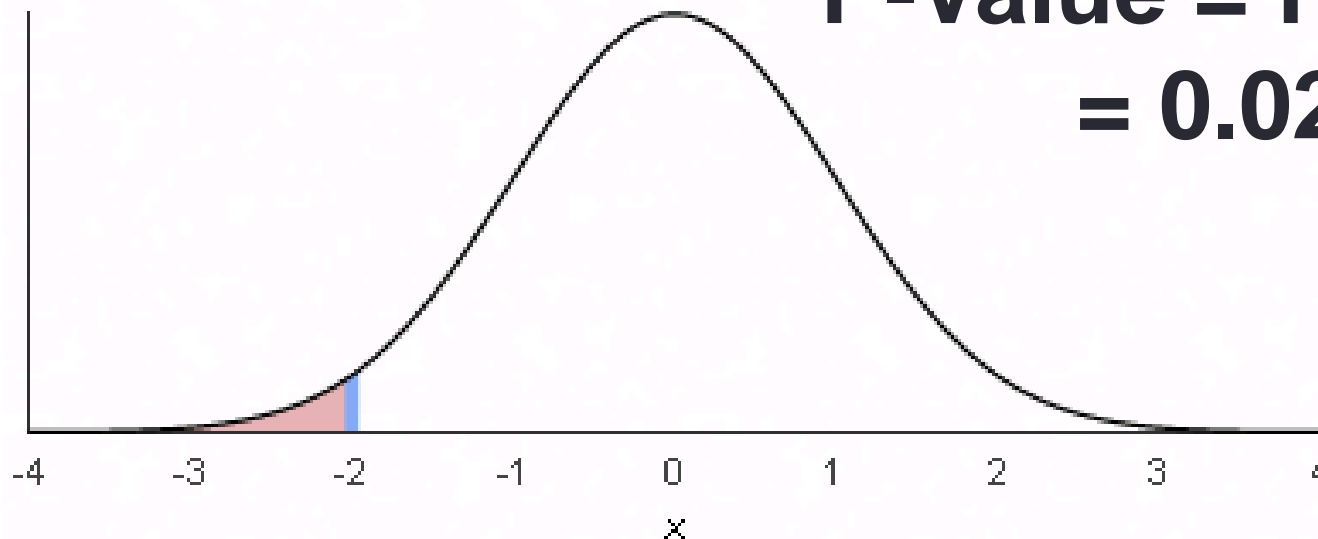
Example 1: Defective Products

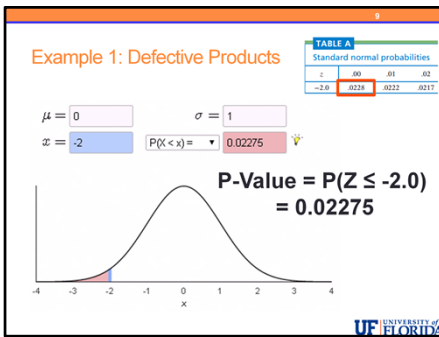
TABLE A			
Standard normal probabilities			
z	.00	.01	.02
-2.0	.0228	.0222	.0217

$$\mu = 0 \quad \sigma = 1$$
$$x = -2 \quad P(X < x) = 0.02275$$



$$\text{P-Value} = P(Z \leq -2.0) = 0.02275$$





Using the normal distribution calculator

(<http://homepage.stat.uiowa.edu/~mbognar/applets/normal.html>)

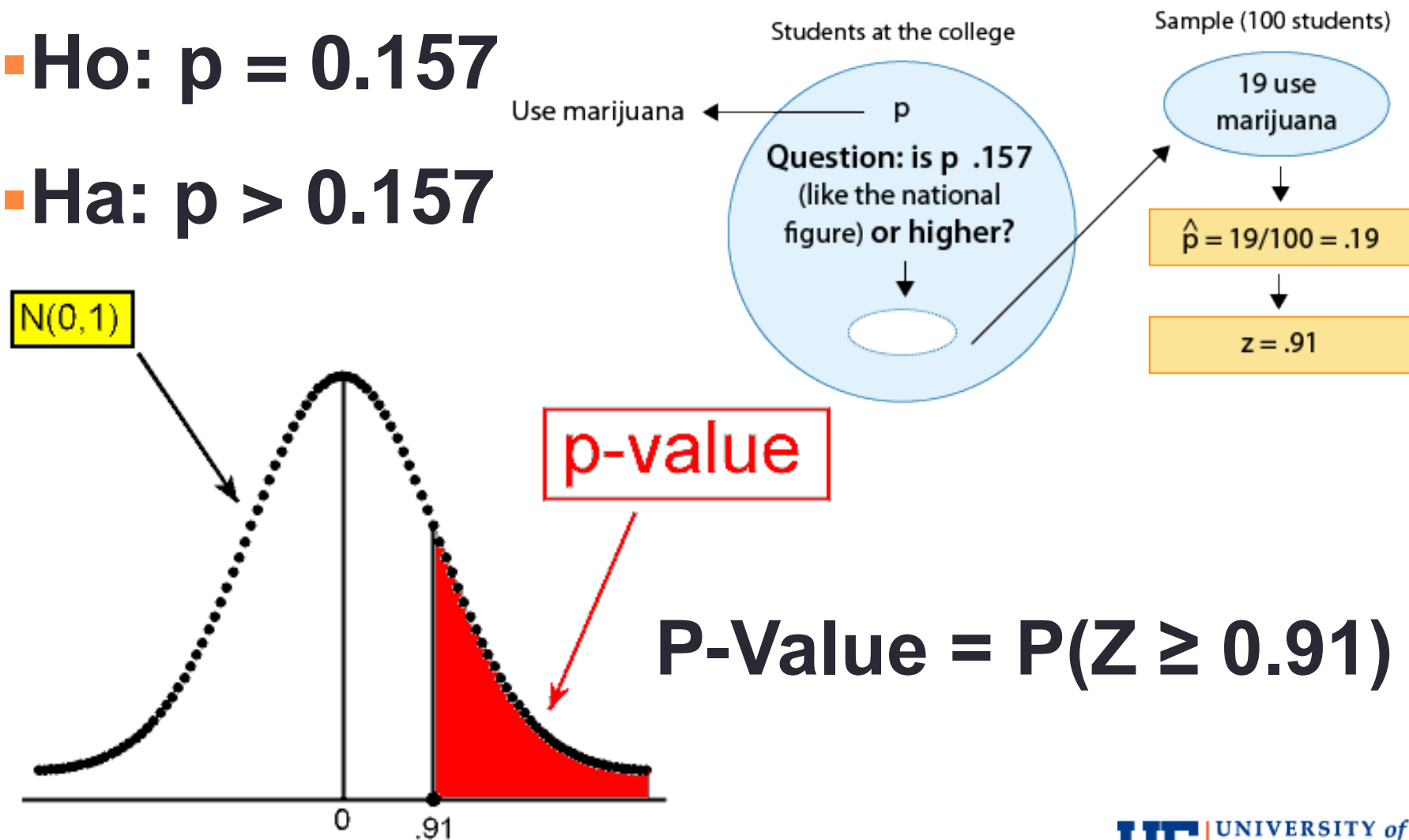
We can enter -2 in for “x” and select $P(X < x)$ from the list to calculate our p-value of 0.02275.

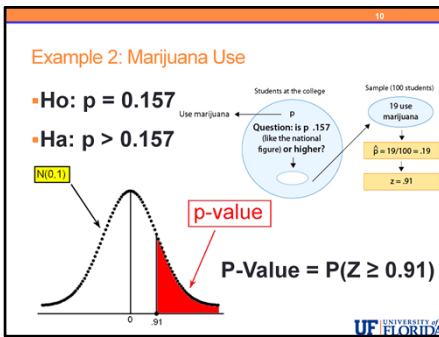
We could also use the normal table and look for the row of -2.0 and the column .00 to find 0.0228.

The p-value tells us that it is pretty unlikely (probability of 0.023) to get data like those observed (test statistic of -2 or less) assuming that H_0 is true.

Example 2: Marijuana Use

- $H_0: p = 0.157$
- $H_a: p > 0.157$





Is the proportion of marijuana users in the college higher than the national figure?

The p-value in this case is:

- The probability of observing a test statistic as large as 0.91 or larger, assuming that Ho is true.

OR (recalling what the test statistic actually means in this case),

- The probability of observing a sample proportion that is 0.91 standard errors or more above the null value ($p_0 = 0.157$), assuming that p_0 is the true population proportion.

OR, more specifically,

- The probability of observing a sample proportion of 0.19 or higher in a random sample of size 100, when the true population proportion is $p_0 = 0.157$

The p-value is found as shown in the normal curve here.



Example 2: Marijuana Use

TABLE A

Standard normal probabilities (continued)

z	.00	.01	.02	.03
0.9	.8159	.8186	.8212	.8238

$$\mu = 0$$

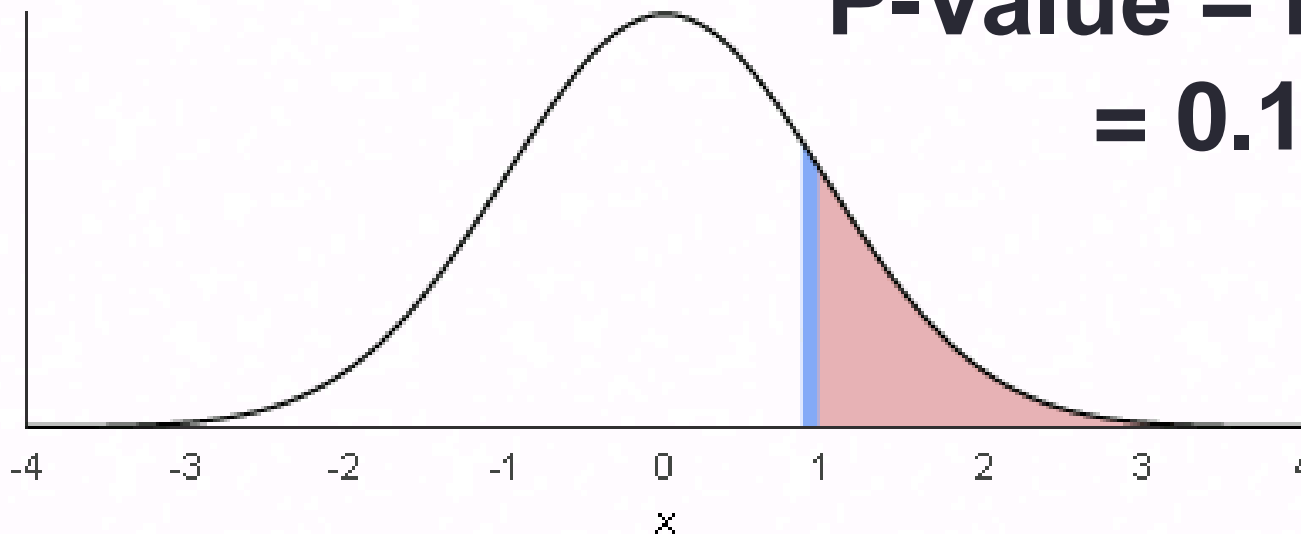
$$\sigma = 1$$

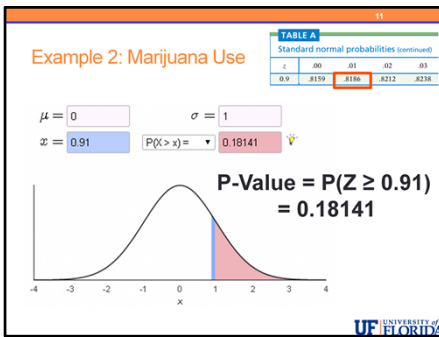
$$x = 0.91$$

$$P(X > x) = 0.18141$$



$$\begin{aligned} \text{P-Value} &= P(Z \geq 0.91) \\ &= 0.18141 \end{aligned}$$





Using the normal distribution calculator
(<http://homepage.stat.uiowa.edu/~mbognar/applets/normal.html>)

We can enter 0.91 in for “x” and select $P(X > x)$ from the list to calculate our p-value of 0.18141

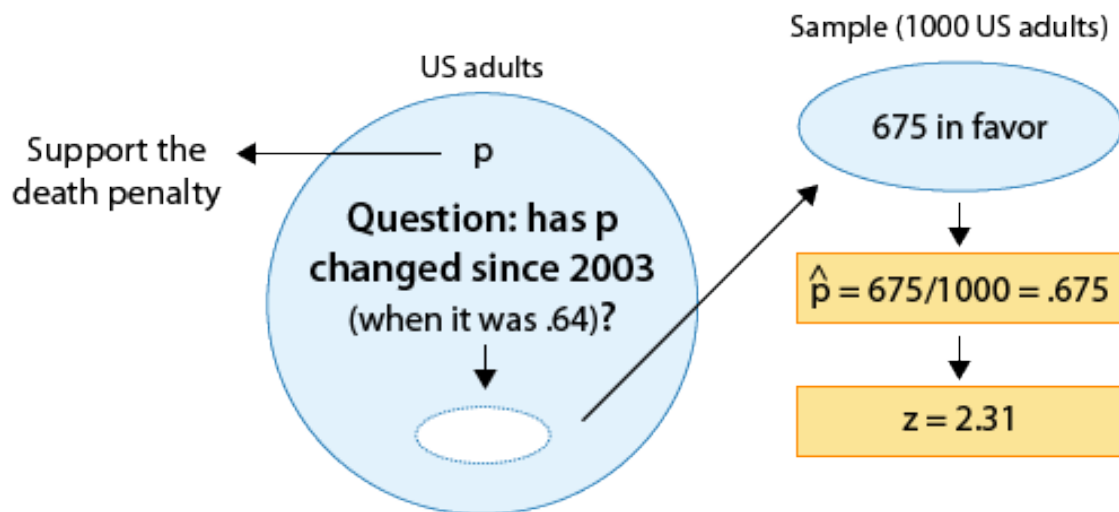
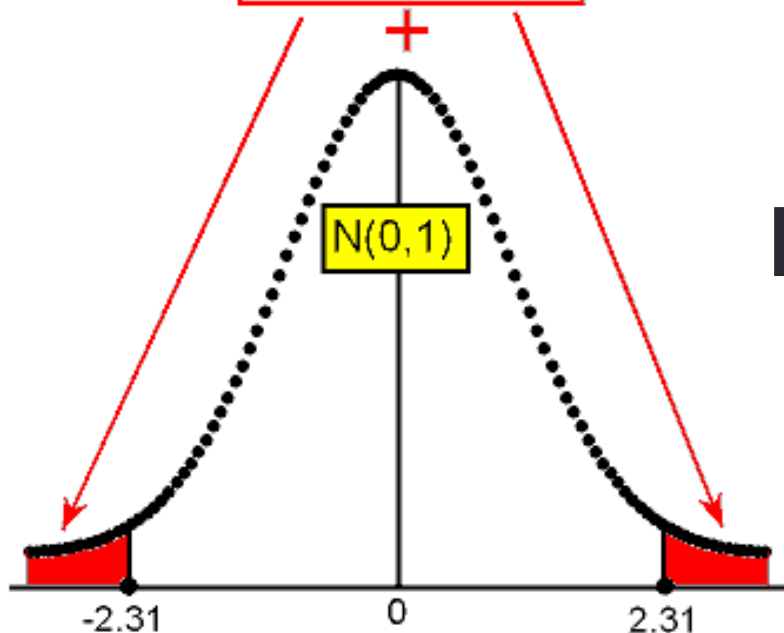
We could also use the normal table and look for the row of 0.9 and the column .01 to find 0.8186, however, this is the $P(X < x)$ and we need the opposite, so if we used the table, we would need to subtract this value 0.8186 from 1 to get 0.1814.

The p-value tells us that it is not very surprising (probability of around 0.18) to get data like those observed (which yield a test statistic of 0.91 or higher) assuming that the null hypothesis is true.

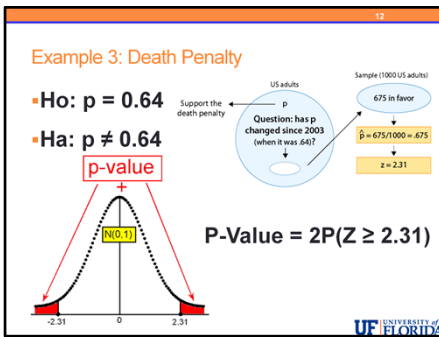
Example 3: Death Penalty

- $H_0: p = 0.64$
- $H_a: p \neq 0.64$

p-value



$$P\text{-Value} = 2P(Z \geq 2.31)$$



Did the proportion of U.S. adults who support the death penalty change between 2003 and a later poll?

The p-value in this case is:

- The probability of observing a test statistic as large as 2.31 (or larger) or as small as -2.31 (or smaller), assuming that H_0 is true.

OR (recalling what the test statistic actually means in this case),

- The probability of observing a sample proportion that is 2.31 standard errors or more away from the null value ($p_0 = 0.64$), assuming that p_0 is the true population proportion.

OR, more specifically,

- The probability of observing a sample proportion as different as 0.675 is from 0.64, or even more different (i.e. as high as 0.675 or higher or as low as 0.605 or lower) in a random sample of size 1,000, when the true population proportion is $p_0 = 0.64$

The p-value is found as shown in the normal curve here. We need both areas – that in the left tail and in the right tail.

Example 3: Death Penalty

TABLE A

Standard normal probabilities (continued)

z	.00	.01	.02	.03
2.3	.9893	.9896	.9898	.9901

$$\mu = 0$$

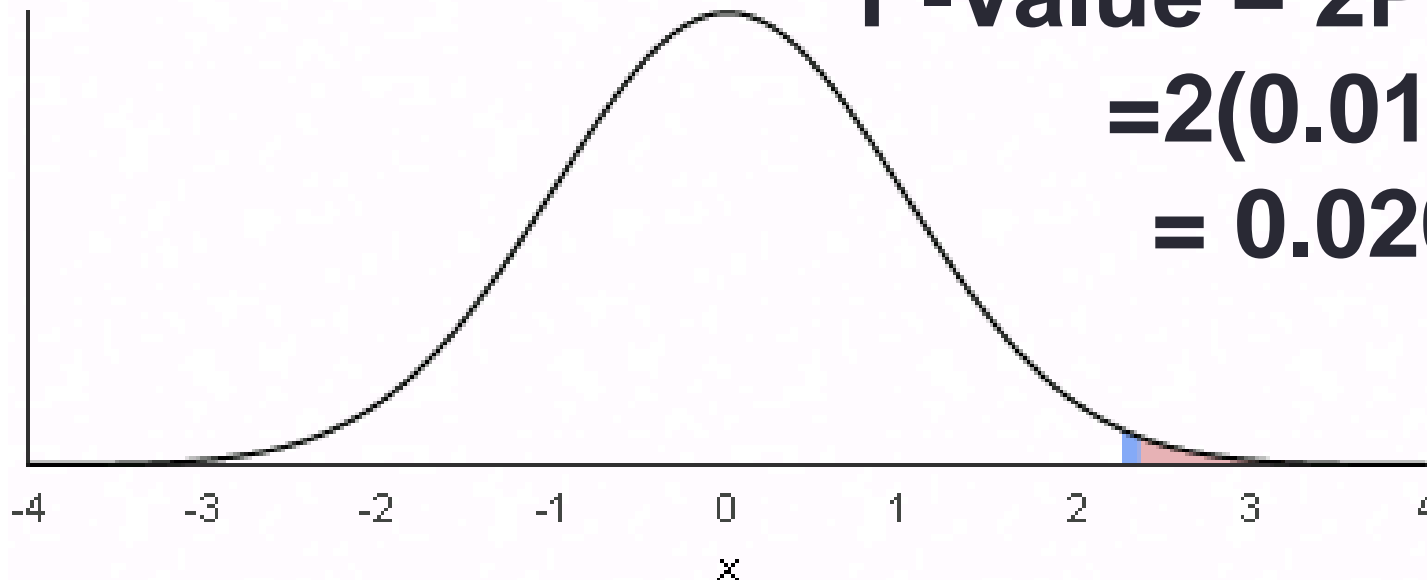
$$\sigma = 1$$

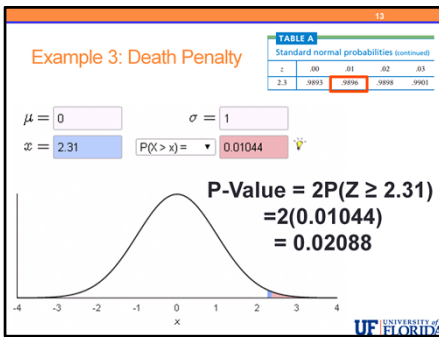
$$x = 2.31$$

$$P(X > x) = 0.01044$$



$$\begin{aligned} \text{P-Value} &= 2P(Z \geq 2.31) \\ &= 2(0.01044) \\ &= 0.02088 \end{aligned}$$





Using the normal distribution calculator

(<http://homepage.stat.uiowa.edu/~mbognar/applets/normal.html>)

We can enter 2.5 in for “x” and select $P(X > x)$ from the list to calculate a probability of 0.01044. This is not yet our p-value however, as for two-sided tests we need both tails – we need to double this value.

$$2 \text{ times } P(Z \geq |2.31|) = 2(0.01044) = 0.02088.$$

We could also use the normal table and look for the row of 2.3 and the column .01 to find 0.9896, however, this is the $P(X < x)$ and we need the opposite, so if we used the table, we would need to subtract this value 0.9896 from 1 to get 0.0104. We would again need to multiply this by 2 to find the p-value in this two sided case.

The p-value tells us that it is pretty unlikely (probability of 0.021) to get data like those observed (test statistic as high as 2.31 or higher or as low as -2.31 or lower) assuming that H_0 is true.




P-VALUES

- Here we have STANDARD NORMAL
- Other COMMON Distributions we will see but NOT work with “BY-HAND” are
 - t -distribution
 - Chi-square distribution
 - F-distribution

14

P-VALUES

- Here we have STANDARD NORMAL
- Other COMMON Distributions we will see but NOT work with "BY-HAND" are
 - t-distribution
 - Chi-square distribution
 - F-distribution



We used this example of one proportion as our test to work through completely by hand because we can use the standard normal distribution, which we have already learned, to calculate p-values.

We will see that many distributions are used to calculate probabilities in practice but for all other situations we will use software to calculate the p-values from our raw data directly.

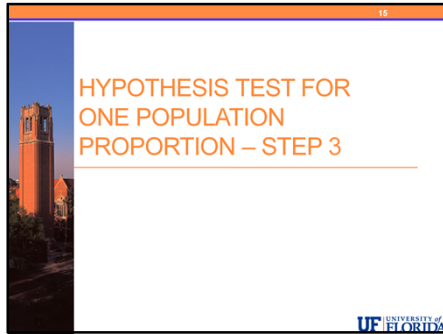
Some common distributions that we will see but will not work with directly are

- t -distribution
- Chi-square distribution
- F-distribution



HYPOTHESIS TEST FOR ONE POPULATION PROPORTION – STEP 3





We are working through the Four Steps in Hypothesis Testing which are

- STEP 1: State the appropriate null and alternative hypotheses, H_0 and H_a .
- STEP 2: Obtain a random sample, collect relevant data, and **check whether the data meet the conditions under which the test can be used**. If the conditions are met, summarize the data using a test statistic.
- STEP 3: Find the p-value of the test.
- STEP 4: Based on the p-value, decide whether or not the results are significant, and **draw your conclusions in context**.

With respect to the z-test the for a population proportion:

Step 1: Completed

Step 2: Completed

Step 3: Completed

Step 4: This is what we will work on next.